REGULAR FIGURES

by

L. FEJES TÓTH

Associate Member
of the Hungarian Academy of Sciences
Professor of Mathematics
University of Veszprém

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THE first part of our book contains some chapters on the classical theory of the regular figures. We start with the Euclidean plane, the "space" most familiar to us, giving a complete enumeration of its discontinuous groups of isometries. We proceed with the analogous problem in spherical geometry (which is the first step towards the far more laborious task concerning Euclidean 3-space). Then we make a journey into the 2-dimensional hyperbolic space and into the general spherical and Euclidean spaces, considering, first of all, their regular divisions.

CHAPTER I
PLANE ORNAMENTS

ONE of the most interesting instances of a deep connection between ar nd mathematics is provided by the surface ornaments, r. d to such an admirable degree of perfection by ancient artists. The task of the artist is to find for a certain type of ornamental symmetry an elementary figure whose repetitions intertwine to give a harmonious whole. The mathematician, in turn, is interested only in the symmetry operations occurring in an ornament. Chapter I deals with the mathematical theory of plane ornaments. In addition, it provides a vivid introduction to one of the most fundamental notions of modern mathematics, the concept of a group.

1. Isometries

An isometry which leaves a figure invariant is called a *symmetry operation*. In order to classify the ornaments according to their symmetry operations we have to investigate the various isometries of the plane.

In the plane, an isometry, i.e. a distance-preserving mapping, is uniquely determined by its effect on a rectangular Cartesian co-ordinate frame. It is said to be *direct* or *opposite* according as it preserves or reverses the sense of the frame. A direct isometry can be achieved by a rigid motion of the plane in itself. Therefore it is often called a *proper motion*. On the other hand, an opposite isometry requires besides a proper motion a reflection in a line. Executing this reflection by a half-turn about the line as axis we obtain, as a final result, a rigid motion in which, however, we must come out of the plane. Therefore an opposite isometry is also called an *improper motion*.

The simplest direct isometries are the *translations*, in which every point of the plane moves through the same distance in the same direction. A translation is uniquely determined by a directed line-segment AB, called a *vector*, leading from a point A to its image B. We shall often denote this translation by $A \to B$. Another type of direct isometry is a *rotation* of the plane through a given angle about a given point. We shall show that no other proper motions exist in the plane.

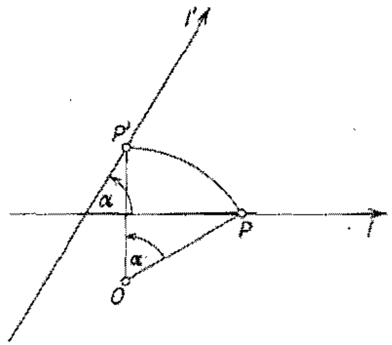


Fig. 1/1

We shall find it convenient to regard translations as rotations through a zero angle about an infinitely distant point. Then our statement reads as follows: *every proper motion of the plane is a rotation.* To make this evident, we note that a proper motion is uniquely determined by a point P, an oriented line l through P and the images P' and l' of P and l. Since the cases where P and P' or the directions of l and l' coincide are trivial, we may suppose that P and P' differ and l and l' include an angle α $(0 < \alpha < 2\pi)$. Let O be a point equidistant from P and P', such that the rotation about O transforming P into P' has an angle equal to α. This rotation transforms l into l' (Fig. 1/1).

The improper motions can also be reduced to a simple type of isometry, called *glide-reflections.* A glide-reflection is the

resultant of a reflection in a line and a translation in the direction of this line. Considering reflections as special cases of glide-reflections, we may assert that *every improper motion of the plane is a glide-reflection.*

In order to show this we notice that an improper motion is determined by the transforms P' and l' of a point P and an oriented line l through it. Consider the line parallel to the bisector of the angle between l and l' passing through the

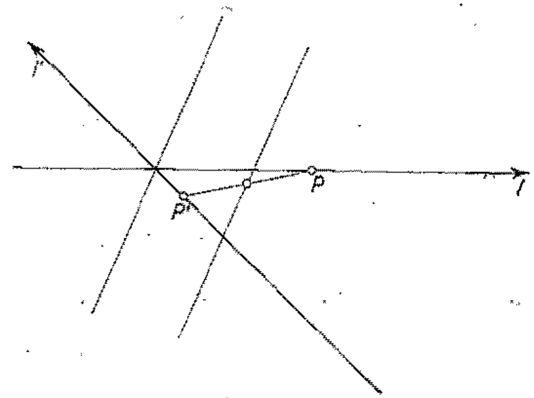


Fig. 1/2

midpoint of the segment PP'. The glide-reflection in this line which carries P into P', transfers l into l' (Fig. 1/2).

Now we make some remarks concerning the composition of isometries. A certain analogy exists between the composition of isometries and the multiplication of numbers. Therefore we denote the transformation arising by performing first the transformation U, then the transformation V, by UV. We call this resultant transformation the *product* of U and V. This kind of multiplication is *associative*:

$$(UV)W = U(VW),$$

so that either side may be denoted by UVW. But it is generally *not commutative*: $UV \neq VU$. If $UV = VU$, (as, for example the reflection and translation in a glide-reflection) we say that U and V *commute*.

We write U^2 for UU, U^3 for UUU, and so on. Furthermore we use the symbol 1 for the identical transformation, which leaves all points invariant, such as, for example, the square R^2 of a reflection R in a line or the cube S^3 of a rotation S through $2\pi/3$. The identity commutes, obviously, with any transformation U: $1U = U1 = U$. Again, we define U^{-1} by $UU^{-1} = 1$. This is the *inverse transformation* of U, which neutralizes the effect of U. We have $(U^{-1}U)U^{-1} = U^{-1}(UU^{-1}) = U^{-1}$. This involves $U^{-1}U = 1$, which expresses the simple fact that each transformation commutes with its inverse.

Now we enunciate a very simple but important fact: the product of two direct isometries or two opposite ones is a direct isometry, while the product of a direct and an opposite isometry (in either order) is always opposite.

Concerning direct isometries, i.e. rotations, we have a simple rule which we shall quote as the *theorem on additivity of angles of rotation*: the product of a rotation through an angle α and a rotation through an angle β is a rotation through the angle $\alpha + \beta$. This becomes evident by taking into consideration the fact that the angle of rotation is given by the change of any oriented line. As an example we note that the product of two rotations about different centres and through equal and opposite angles is a translation, because the resultant rotation has the angle zero, but cannot be the identity since neither of the two centres of rotation remains unchanged.

The following more general *rotation product theorem* also gives information on the centre of rotation of the product of two non-degenerate rotations. Let $\alpha/2$, $\beta/2$ and $\gamma/2$ be the external or internal angles of a triangle ABC according as the vertices are named in the positive or negative sense. Then the product of rotations through angles α, β and γ about A, B and C is the identity.

Hence the product of the rotations through α and β about A and B is a rotation about C through $-\gamma \equiv \alpha + \beta$ (mod 2π).

To prove this, we consider the product of the reflections in the lines AC and AB. This is a proper motion leaving A fixed and turning AC about A through α. Hence it is a rotation through α about A. Similarly, replacing the rotations about B and C each by two reflections, the product of the rotations can be expressed as the product of the reflections in the lines AC, AB, BA, BC, CB, CA. But this product is, obviously, the identity.

We shall often make use of the product $U^{-1}VU$, called V *transformed by* U. For instance, if S is a rotation (spin) about A through the angle α and P an arbitrary proper motion transforming A into A' then $P^{-1}SP$ is a rotation about A' through α. In fact, it is, as a product of direct isometries, itself direct, i.e. a rotation. By the theorem of the additivity of angles this rotation has an angle α. Finally, the centre of rotation must be A', because P^{-1} transforms A' into A, S leaves A invariant, and P transforms A back into A'.

Similarly, it can be seen that, if the improper motion I transforms the centre A of the rotation S of angle α into A' then $I^{-1}SI$ is a rotation about A' of angle $-\alpha$.

As a further example consider a translation T transformed by a rotation S of angle α: $S^{-1}TS$. This is a translation whose magnitude is the same as that of T and whose direction forms an angle α with the direction of T. Indeed, $S^{-1}TS$ is a proper motion, namely a translation, its angle, by the additivity theorem, being zero. Let A' be the image of the centre A of the rotation S under T and A'' the image of A' under S. All that remains to be proved is that the translation $S^{-1}TS$ moves A into A''. But this is true because S^{-1} leaves A unchanged, T transforms it into A' and S moves A' into A''.

It can also be seen that if G is a glide-reflection then $G^{-1}TG$ is a translation of the same magnitude as T, the direction of which arises from the direction of T by reflection in the axis of G. Recapitulating the two cases: if T is a translation and U any isometry then $U^{-1}TU$ is a translation whose vector is nothing else but the image of the vector of T under U.

2. Symmetry Groups

A set is called a *group* if an associative and invertible multiplication is defined in it. (An operation is said to be invertible if its effect can always be neutralized.) More precisely, the defining properties of a group are: (i) there is an operation, called multiplication, which assigns to every two of its elements U and V an element of the group, called the product of U and V and denoted by UV; (ii) for every three elements U, V, W we have $U(VW) = (UV)W$; and (iii) each element U has an inverse U^{-1} in the group, such that for each element V we have $UU^{-1}V = V$.

It is then easily deduced that $VUU^{-1} = V$ and $UU^{-1} = U^{-1}U$. We write $UU^{-1} = 1$ and call it the unit element of the group.

Obviously, the totality of the symmetry operations of a figure constitutes a group, with respect to the composition of its transformations as group operation. This group is called the *symmetry group* of the figure. It may happen that the symmetry group of a figure consists of the identity only. Then the figure is said to be *asymmetrical*. In all other cases the figure is called *symmetrical*.

Conversely, a given group of at least two isometries always determines a symmetrical figure consisting of the images of a certain "elementary figure" under the transformations of the group. These images are indiscernible from one another, both with respect to their shapes and sizes, as well as to their mutual positions. Therefore they are said to be *equivalent* under the group.

The notion of the symmetry group will soon prove to be a powerful tool in surveying the vast family of symmetrical figures. One essential element in this concept-building is abstraction: we do not concern ourselves with the special kind of figures, merely to the totality of their symmetry operations. On the other hand, we have noted the features of primary importance of such a totality: it contains the inverse of each of its operations as well as the product of every two of its operations.

Together with the notion of symmetry we have to explain the usage of the word "regular". The words "regular" and "symmetrical", as applied to geometrical figures, have similar meanings; but the former is not so unambiguous as the latter. Regularity generally denotes a little more than symmetry and it may be considered, roughly expressed, as a symmetry of higher degree.

Usually, we apply the word "regular" to configurations consisting of different kinds of constituents, as, for example, the vertices, edges and faces of a polyhedron. In this case regularity requires regular arrangement, i.e. equivalence under a group of isometries, of all kinds of components, or else regularity (in a certain sense) of the component elements, or both. So regularity must be defined in each case separately. But in the case of a single kind of regularly arranged elementary figures the whole figure may equally be called regular and symmetrical.

We shall concern ourselves only with discrete groups of isometries. A group of transformations is said to be *discrete*, or *discontinuous*, if every point has a discrete set of transforms, i.e. if every point has a neighbourhood containing none of its transforms save the given point itself. A discrete set of points may have a point of accumulation. But it is easy to see that the points equivalent under a discrete group of isometries can accumulate nowhere.

We obtain a vivid insight into the structure of such a group by a connected region, called *unit cell*, or *fundamental region*, whose transforms cover the plane without overlapping and without gaps. This notion enables us to divide the discrete groups of isometries into two classes, according as the unit cell of the group is infinite or finite. By analogy with 3-space, the latter are called, in any number of dimensions, *crystallographic groups*.

Our discussions will be facilitated by the important notion of the *subgroup*. This is a subset of the group which itself constitutes a group. For instance, in view of the fact that the product of two translations is itself a translation, the set of all translations contained in the symmetry group of a figure is a subgroup of the original one. It is called the *translation-group* of the figure. Similarly, we speak of the *rotation-group* of a figure, i.e. of the set of all direct isometries present in its symmetry group. But these subgroups of simpler structure must occur separately in our enumeration. This allows us to start with groups of simpler type and to progress gradually to more complicated ones.

The above classification of our groups by the unit cell can also be effected by the subgroups of the translations contained in them. We shall see that in the plane the groups with infinite unit cells may be defined as groups containing either no translations at all or, at most, parallel translations. On the other hand, the crystallographic groups of the plane are characterized by their containing non-parallel translations.

In both classes we shall consider first the groups of proper motions. The remaining groups of isometries can be constructed with the help of the following:

Note. If the proper motions contained in a group of isometries are

$$P_1,\ P_2,\ \ldots$$

and I is an improper motion of the group, then all improper motions of the group are given by

$$IP_1,\ IP_2,\ \ldots .$$

In fact, the products IP_1, IP_2, $\ldots$, each being the product of an opposite and a direct isometry, are opposite. They all belong to the group, by the group postulate (i). On the other hand, let $\bar{I}$ be an arbitrary opposite isometry in the group. Then, on account of the group postulates (i) and (iii), $I^{-1}\bar{I}$ is an element of the group. Being the product of two opposite isometries, it is direct and therefore $I^{-1}\bar{I} = P_i$, for some index i. Consequently we have, in accordance with our note

$$\bar{I} = II^{-1}\bar{I} = IP_i.$$

The above note offers a suitable point at which to introduce some further fundamental notions. If a subgroup $\mathfrak{S}$ of a group $\mathfrak{G}$ consists of S_1, S_2, $\ldots$, while G is any element of $\mathfrak{G}$, the set $G\mathfrak{S}$ of elements GS_1, GS_2, $\ldots$ is called a left *coset* of $\mathfrak{S}$, and the set $\mathfrak{S}G$ consisting of S_1G, S_2G, $\ldots$ is called a right coset. Consider another left coset of $\mathfrak{S}$ consisting of HS_1, HS_2, $\ldots$, and suppose that the two cosets contain identical elements $GS_i = HS_j$. Then, for any element S_k, $GS_k = GS_iS_i^{-1}S_k = HS_jS_i^{-1}S_k = HS_l$ (for some index l), showing that the two cosets are identical, apart from the succession of elements. Thus any two left (or right) cosets have either the same elements or entirely different ones. This circumstance allows a distribution of the elements of $\mathfrak{G}$ into a certain number of distinct left cosets. This number (which may be infinite) is called the *index* of the subgroup $\mathfrak{S}$. For instance, the index of the subgroup of proper motions in a group of isometries containing improper motions always equals 2. Here the two cosets are none other than the set of direct and opposite isometries, respectively.

The *order* of a finite group is defined as the number of elements contained in the group. If a group of order n contains a subgroup of order k and index i, we obviously have $n = ik$, showing that for finite groups the index may be defined as the quotient of the orders of the group and subgroup. Our considerations imply a fundamental theorem of Lagrange, according to which this quotient is always an integer.

3. Groups with Infinite Unit Cells

We shall call the discrete groups of isometries of the plane *ornamental groups*. We start with the ornamental groups free from translations. We shall call them *rosette groups*. They are sometimes known as *point groups*, for the following reason.

Such a group contains only rotations about a single centre. For, if there were two rotations S_1 and S_2 with distinct centres O_1 and O_2, the transformation $S_1^{-1}S_2^{-1}S_1S_2$ of the group would be a degenerate rotation, by the additivity theorem for angles of rotation. It cannot be the identity, since it displaces O_1 into the image of O_1 under $S_2^{-1}S_1S_2$, i.e. under S_1 transformed by S_2. But the centre of this transformed rotation is the image of O_1

under S_2, i.e. a point different from O_1 and therefore $S_2^{-1}S_1S_2$ effects a change in O_1. Hence $S_1^{-1}S_2^{-1}S_1S_2$ would be a non-degenerate translation, contrary to our assumption.

Let us consider first the case in which the group contains direct isometries only, i.e. rotations about a single centre O. Since such a rotation is uniquely given by the image of a point A_1 different from O, it suffices to know the set of the transforms of A_1. These transforms lie on the circle with centre O passing

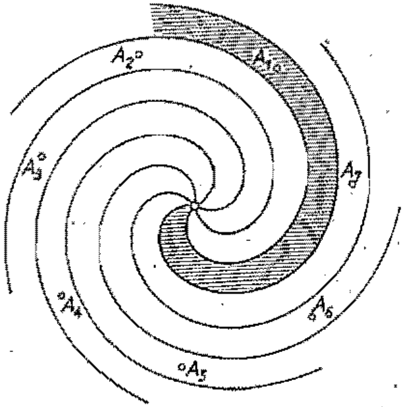


Fig. 3/1

through A_1. Owing to the discontinuity of the group there is only a finite number, p, of points which we denote in their cyclical order by $A_1, A_2, \ldots, A_p$. The rotation transferring A_1 into some other point A_i, transforms the set $A_1, \ldots, A_p$ into itself. Hence this rotation transforms A_2 into A_{i+1} ($A_{p+1} \equiv A_1$) and therefore $\sphericalangle A_1OA_2 = \sphericalangle A_iOA_{i+1} = 2\pi/p$. This is the smallest angle of rotation in the group and the others are multiples of it. In other words, denoting the rotation about O through angle $2\pi/p$ by S, all operations of the group are $S, \ S^2, \ldots, S^p = 1$. Further powers of S are merely repetitions of the listed ones, e.g. $S^{p+1} = S$, $S^{-1} = S^{p-1}$, etc.

The centre O is called a *centre of p-fold rotation*, or a p-gonal (di-gonal, tri-gonal, etc.) centre. We shall also use the terms *diad*, *triad* and so forth.

In order to construct a unit cell, let us consider a curve having exactly one point in common with each circle centred at O. The region swept over by this curve by turning it about O through $2\pi/p$ will be a unit cell (Fig. 3/1). The simplest unit cell arises by choosing for the curve a ray emanating from O, obtaining as unit cell an angular region with vertex O and angle $2\pi/p$. Fig. 3/2 shows that even more general unit cells may be formed to our group.

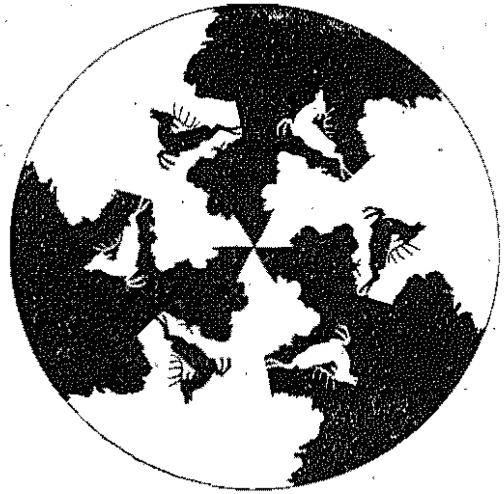


Fig. 3/2

Usually, a rotation group of a figure is called after the figure itself. Therefore we call the group just described (which is the rotation group of a regular p-gon) the *polygonal group of order p*. We shall denote it by $\mathfrak{c}_p$. The group $\mathfrak{c}_1$ consists of the identical transformation only; its unit cell is the whole plane.

Several of these groups are illustrated by the transforms of an asymmetrical spiral, as an elementary figure (Fig. 3/3). In some cases it is placed in two distinct positions in the unit cell, in order to demonstrate the different artistic effects they produce.

Let us now consider the rosette groups which also contain opposite isometries. The direct isometries contained in such a group form a subgroup $\mathfrak{c}_p$ and the opposite isometries can only be reflections in lines passing through the centre O of the rotations $S, S^2, \ldots, S^p$ of $\mathfrak{c}_p$. In fact, if U is any transformation of the group then $U^{-1}SU$ is a non-degenerate rotation about the image O' of O under U. But O and O' must coincide and this proves our assertion.

This is a special case of a general fact to which we shall refer as the *principle of invariance of the centres of rotation* under transformations of the group. It states that any transformation of an ornamental group transforms the set of the centres of rotation of the group (together with their multiplicities) into themselves.

In virtue of the note in Section 2 the opposite transformations of the group are $R, RS, \ldots, RS^{p-1}$, where R is a

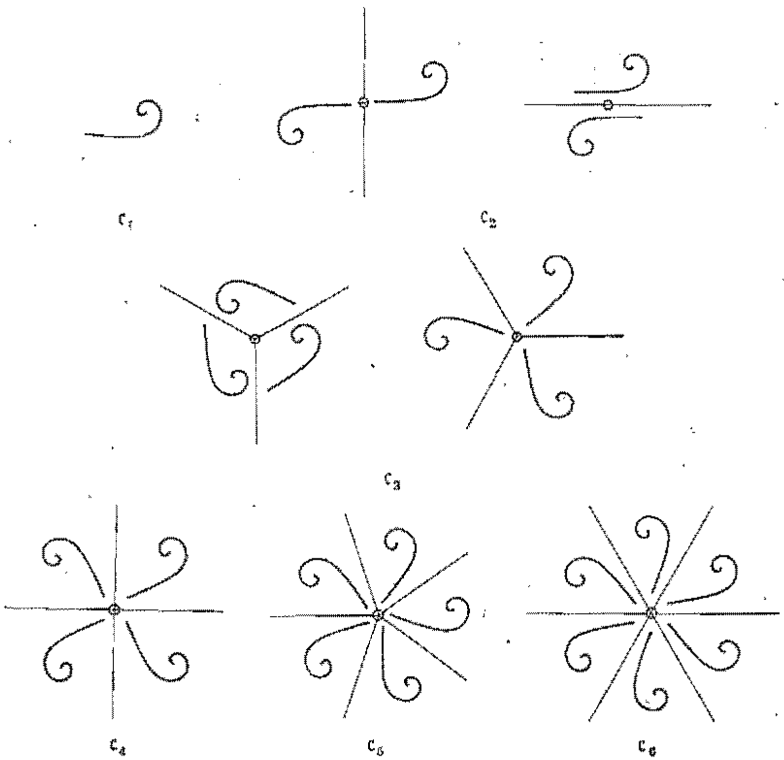


Fig. 3/3

reflection in a line l through O. But since the reflection RS^i transforms l into the image l_i of l under S^i, it must be a reflection in the bisector of the angle formed by l and l_i. Consequently, the opposite transformations of the group are reflections in p lines which divide the plane into $2p$ equal angular regions. A unit cell of the group is furnished by one of these regions (Fig. 3/4). We shall denote this group by $\mathfrak{d}_p$. It is the symmetry group of a regular p-gon. Its order is $2p$.

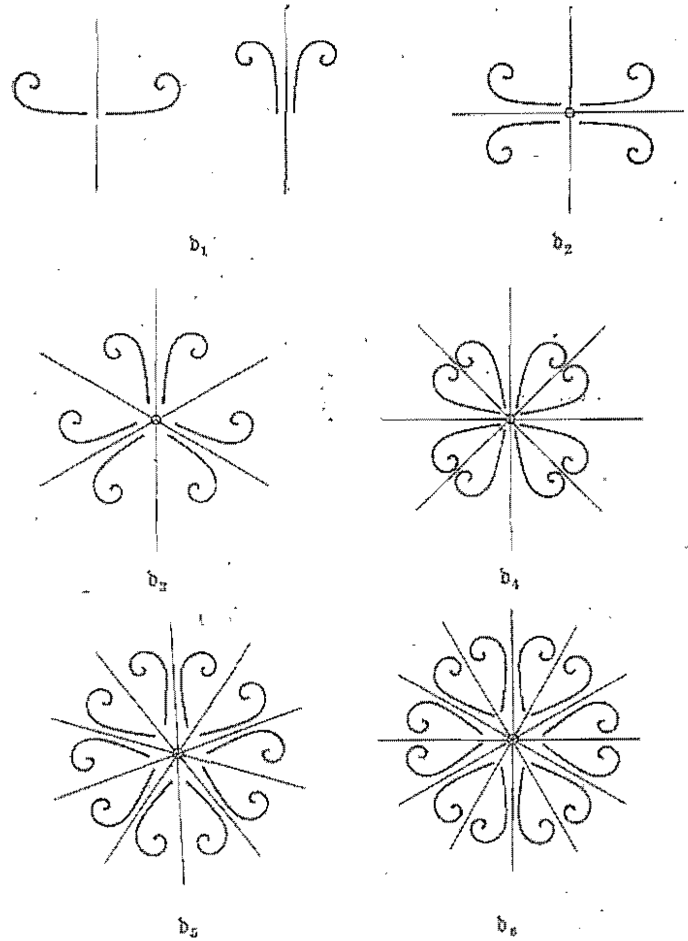
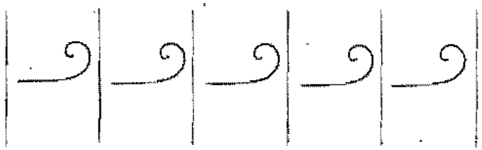


Fig. 3/4

This completes the enumeration of the rosette groups. These are the only ornamental groups of finite order.

We now turn to the *frieze groups*, i.e. to ornamental groups containing only unidirected translations (and their inverses). We begin with the simplest group of this type which consists entirely of translations. Then the transforms of a point must lie on one line. In view of the discontinuity, these transforms divide the line into segments. But these segments must be equivalent under the group; consequently they are congruent. Denoting the translation of the group through the distance of such a segment by T, all the transformations in the group are $\ldots, T^{-2}, T^{-1}, 1, T, T^2, \ldots$. A simple unit cell is a strip which cuts our line in one of the above segments. We shall denote this group, known as the one-dimensional translation group, by $\mathfrak{F}_1$ (Fig. 3/5).

We proceed with the case in which the group also contains



$\mathfrak{F}_1$.
Fig. 3/5

rotations, but does not contain improper motions. If S is a rotation of angle α then a translation of the group transformed by S is a translation including the angle α with the original one. Hence $\alpha = \pi$ and S must be a half-turn. Thus the group admits only di-gonal centres of rotation. We proceed to determine their totality.

If A_0 is a diad, then its successive transforms $A_1, A_2, \ldots$ and $A_{-1}, A_{-2}, \ldots$ under the smallest translation T of the group and its inverse T^{-1} are also diads. Consider the product TS_1, where S_1 is a half-turn about A_1. It is itself a half-turn by the additivity theorem and, since it transforms A_0 into A_1, its centre must be the midpoint B_0 of the segment A_0A_1. Similarly, all midpoints

B_i of the segments A_iA_{i+1} $(i = 0; \pm 1, \pm 2, \ldots)$ must be di-gonal centres of the group. On the other hand, there are no further diads in the group. If, for example, S is any half-turn of the group having the centre O, and S_0 is the half-turn about A_0, then S_0S is a translation and therefore it transforms A_0 into some point A_n. But, S_0 leaving A_0 invariant, S also must transform A_0 into A_n. Therefore O, being half-way between A_0 and A_n, must be one of the diads A_i or B_i (Fig. 3/6).

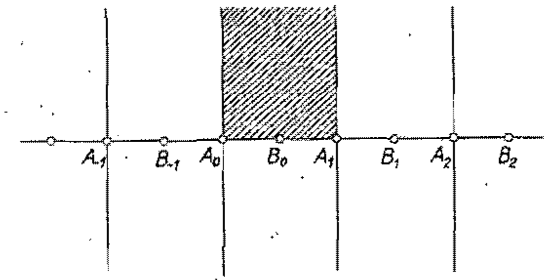
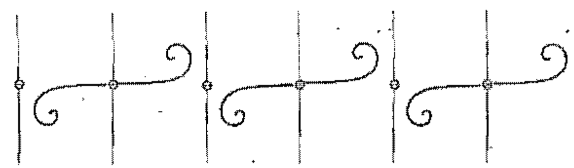


Fig. 3/6

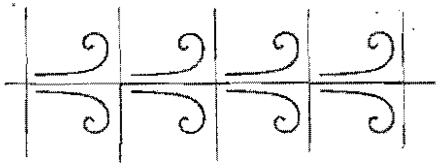
This completes the discussion of the group. As unit cell we may choose a half-strip based upon the segment A_0A_1. We denote this group by $\mathfrak{F}_2$ (Fig. 3/7).



$\mathfrak{F}_2$
Fig. 3/7

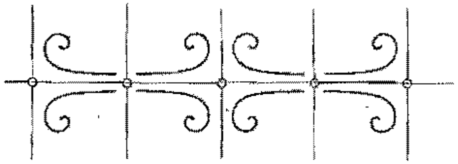
Now we try to join to the two groups discussed above an opposite isometry. Let R be a reflection in a line l, and T the smallest translation in a frieze group. Then $R^{-1}TR$ is a translation, the vector of which is the image of the vector of T under R. Therefore l must be either parallel or orthogonal to the direction of T.

First we deal with the case when R is a reflection in a line parallel to T. Joining the products RT^i to the translations T^i ($i = 0, \pm 1, \pm 2, \ldots; T^0 = 1$) of $\mathfrak{F}_1$, we obtain a group which we shall denote by $\mathfrak{F}_1^1$ (Fig. 3/8). For, in view of $RT^i = T^iR$ and $R^2 = 1$, we have for any integers i and j $RT^iRT^j = T^{i+j}$ and $RT^iT^j = T^iRT^j = RT^{i+j}$. Again, $(RT^i)^{-1} = RT^{-i}$, whereby the group postulates are verified.



$\mathfrak{F}_1^1$
Fig. 3/8

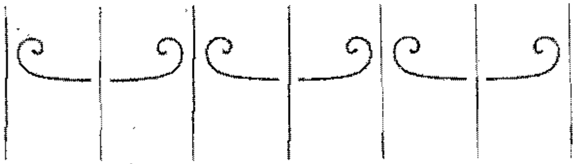
Endeavouring now to enlarge the group $\mathfrak{F}_2$ in a similar manner, we find (by the principle of invariance of the centres of rotation) that the line of reflection (parallel to T) can only be the line l of the diads. If R is the reflection in the line l, the set formed by the transformations of $\mathfrak{F}_2$ and their premultiples (or postmultiples) by R, constitutes a group $\mathfrak{F}_2^1$ (Fig. 3/9).



$\mathfrak{F}_2^1$
Fig. 3/9

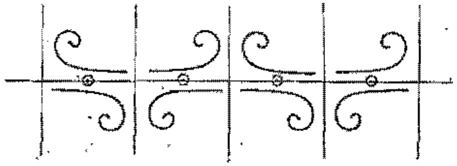
If S is a half-turn of $\mathfrak{F}_2$ of centre A then RS is a reflection in the line through A perpendicular to l. Therefore the group $\mathfrak{F}_2^1$ contains besides R a set of reflections in lines passing through the diads perpendicular to l.

Now let R be a reflection in a line r perpendicular to the direction of T. It can be seen immediately that the transformations T^i, RT^i ($i = 0, \pm 1, \pm 2, \ldots$) form a group, which we denote by $\mathfrak{F}_1^2$. The product RT^i is a reflection in a line parallel to r which bisects the strip bounded by r and its image under T^i. So $\mathfrak{F}_1^2$ contains, together with r, a row of parallel and equidistant lines of reflection, the distance between consecutive lines being the half distance of T (Fig. 3/10).



$\mathfrak{F}_1^2$
Fig. 3/10

We now attempt to enlarge the group $\mathfrak{F}_2$ by a reflection R in a line r perpendicular to the line of the diads. If r passes through a diad the resulting group is, by virtue of the remark of Section 2, identical with $\mathfrak{F}_2^1$. Thus we may suppose that r contains no centres of rotation. Then, owing to the invariance of the diads under R, the line r must bisect the segment determined by two neighbouring diads. Under these conditions we obtain a group which we denote by $\mathfrak{F}_2^2$ (Fig. 3/11).

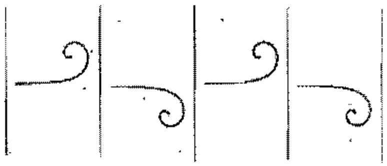


$\mathfrak{F}_2^2$
Fig. 3/11

In this way the possibilities arising by reflections are settled. All we need is to scrutinize the case of glide-reflections.

If G is a glide-reflection, G^2 is a translation and we have for some integer $n \geqslant 1$ either $G^2 = T^{2n}$, or $G^2 = T^{2n-1}$, where T is one of the smallest translations of the group. In the first case

GT^{-n} is a reflection and we are led to the groups $\mathfrak{F}_1^1$ or $\mathfrak{F}_2^1$. If, on the other hand, $G^2 = G^{2n-1}$, then GT^{-n+1} is a glide-reflection such that $(GT^{-n+1})^2 = T$. Changing the name of GT^{-n+1} to G, we have $G^2 = T$. It may be seen immediately that the translations and glide-reflections T^i, GT^i $(i = 0, \pm 1, \ldots)$ form a group. We denote this group, being the third group in our discussion derived from $\mathfrak{F}_1$, by $\mathfrak{F}_1^3$ (Fig. 3/12).



$\mathfrak{F}_1^3$
Fig. 3/12

Now let G be a glide-reflection of a frieze group containing $\mathfrak{F}_2$ as subgroup. Then the axis of G must coincide with the line l of the diads. Therefore, if S is a half-turn of $\mathfrak{F}_2$ then GS is a reflection in a line perpendicular to l, and we arrive at the group $\mathfrak{F}_2^2$ constructed previously.

We have now constructed all possible frieze groups. We have found that there are seven such groups. Restricting ourselves to groups containing merely direct transformations this number drops to two.

Let us look again at our notation. $\mathfrak{F}_p$ is a group of direct transformations containing at most p-gonal centres of rotation and $\mathfrak{F}_p^1, \mathfrak{F}_p^2, \ldots$ are the groups containing opposite transformations; the subgroup of direct transformations in each is $\mathfrak{F}_p$. It is a lucky accident that this principle can also be used for the wall-pattern groups to be discussed in the next section. In 3-space, however, an analogous notation would break down owing to the fact that there are various crystallographic groups of direct transformations here, having at most, say, 3-fold axes of rotation.

All groups discussed in the present paragraph have infinite unit cells. In the next paragraph we shall construct the remaining ornamental groups, finding that their unit cells are finite. This will involve an indirect proof of the completeness of the above enumeration of the ornamental groups with infinite unit cells.

4. Groups with Finite Unit Cells

In this section we shall complete the enumeration of the ornamental groups by the *wall-pattern groups*, i.e. by the ornamental groups containing two non-parallel translations. To begin with we shall construct the wall-pattern groups of direct transformations, treating, first of all, the simplest type, namely those free from rotations.

Let S be a complete set of equivalent points (the set of points equivalent to a point) under the group in question. Clearly, there are in S three non-collinear points A, B, C such that the triangle ABC does not contain besides A, B and C further points of S. Let the translation $B \to C$ move A into D. The parallelogram $ABCD$ generates a *point lattice*, viz. the set of the vertices of all parallelograms which arises from $ABCD$ by the translations $A \to B$, $B \to C$, $C \to D$, $D \to A$ and their repeated applications.

Obviously, all points of this lattice belong to S. On the other hand, S does not contain additional points. For, since the above parallelograms cover the plane without interstices, such a point must lie in (or on the boundary of) a parallelogram. Then, in view of the equivalence of the parallelograms, $ABCD$ would also contain a point P of S different from the vertices A, B, C and D. Since the point P cannot lie in ABC (by the supposition made about this triangle), it must be contained in ACD. But then the translation $P \to D$ of the group would move B into a point P' of ABC and the triangle ABC would contain, after all, the point P' of S (Fig. 4/1). This contradiction proves our assertion.

Hence all transformations of the group are $T_1^i T_2^j$ $(i, j = 0, \pm 1, \ldots)$ denoting by T_1 and T_2 the translations $A \to B$ and $B \to C$. This group is known as the (two-dimensional) lattice group. The simplest unit cell is a generating parallelogram, such as $ABCD$. We denote this group by $\mathfrak{W}_1$ (Fig. 4/2).

We now stipulate that the group contains a half-turn but no other kind of rotation. The subgroup of translations $\mathfrak{W}_1$ contained in the group transforms a diad A of the group into the points of a lattice; all these points are likewise diads. Now we can refer

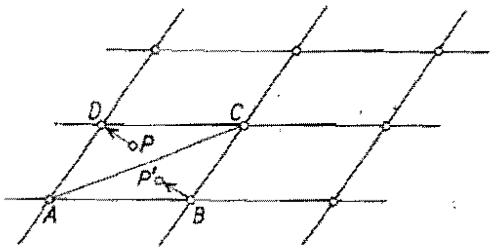
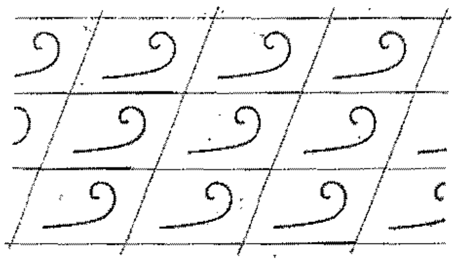


Fig. 4/1



$\mathfrak{W}_1$

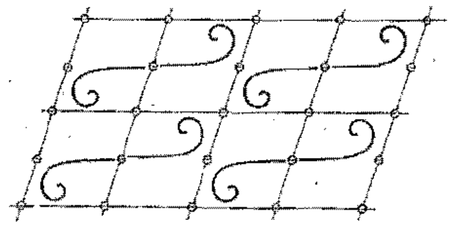
Fig. 4/2

to the discussion of the group $\mathfrak{F}_2$, according to which the centroid of every two lattice points must also be a diad and, conversely, every diad of the group must be midway between two lattice points. Thus the centre, the vertices and the midpoints of the sides of a generating parallelogram provide for the totality of diads.

We are dealing with all translations and half-turns which leave a point lattice invariant. But these transformations, evidently, form a group. Its symbol is, by our notation, $\mathfrak{W}_2$. As unit cell we can choose one of the triangles into which a diagonal splits a generating parallelogram or another suitable half part of a generating parallelogram (Fig. 4/3).

The discussion of the remaining cases follows, mainly, in a similar way. First we shall determine in each case the arrangement of the centres of rotation.

Let P be a p-gonal centre of rotation, such that $p > 2$. The group being discrete, there is a least distance from P at which we can find another centre Q of more than 2-fold rotation, say of q-fold rotation. We consider the product of the rotation about P through $2\pi/p$ and the rotation about Q through $2\pi/q$. This



$\mathfrak{W}_2$

Fig. 4/3

is, according to the rotation product theorem of Section 1, a rotation about a point R through $-2\pi/r$ such that the triangle PQR of negative circuit has angles π/p, π/q, π/r. The angle sum of a triangle being π, we have

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

Consider first the case $p = q = 3$. This implies $r = 3$, in consequence of which we have a set of triads forming a lattice of equilateral triangles (i.e. a point-lattice having a generating parallelogram composed of two equilateral triangles). Apart from these triads the group contains no other centres of rotation. To show this we observe first that the six triangles meeting at P do not contain any centre of more than 2-fold rotation, except the vertices. Then, in view of the fact that any triangle can be transformed by a third-turn about the triads into one of the six triangles, the same holds for every triangle. This means that the statement is verified for centres of more than 2-fold rotation.

Diads cannot occur either, because a rotation through π together with a rotation through $-2\pi/3$ would result in a rotation through $2\pi/6$, which is impossible.

Now besides $p \geqslant 3$ and $q \geqslant 3$, we stipulate $p+q \geqslant 6$. Then $r<3$, and consequently $r<q$. Thus the triangle PQR has a smaller angle at Q than at R, in consequence of which R lies nearer to P than Q. It follows that R is a diad, and since $r=2$, we have

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{2}.$$

This equation allows two further cases only, namely $p=q=4$ and $p=3$, $q=6$ (or, which is the same thing, $p=6$, $q=3$). This completes the proof of an important fact, known as the *crystallographic restriction*: 5-fold and more than 6-fold centres of rotation are inadmissible in infinite ornamental groups.

The following treatment applies to both cases mentioned above. Successive rotations through $2\pi/p$ about P transform Q into the vertices of a regular p-gon and successive rotations through $2\pi/q$ about Q transform this p-gon into a set of qp-gons. Further rotations through $2\pi/q$ about the old and the newly obtained vertices lead to a *regular tessellation* $\{p, q\}$, i.e. to a set of regular p-gons, q at each vertex, fitting together side by side to cover the whole plane simply and without gaps (Fig. 4/4). The face-centres are p-fold, the vertices q-fold and the midpoints of the sides 2-fold centres of rotation. These centres of rotation are responsible for all rotations belonging to the group. In showing this we can restrict ourselves to the p-gon of centre P with which we started. It contains, by the initial supposition, no centres of more than 2-fold rotation other than those mentioned. In order to show the same for di-gonal centres, let us take into account that a diad cannot lie other than midway between P and another p-gonal centre. Thus the only problematical points are the midpoints of the segments joining P with the vertices, if $p=q=4$. But a rotation through $2\pi/4$ about P composed with a half-turn about the centroid of P and Q would give a rotation through $-2\pi/4$ about R, which is inadmissible.

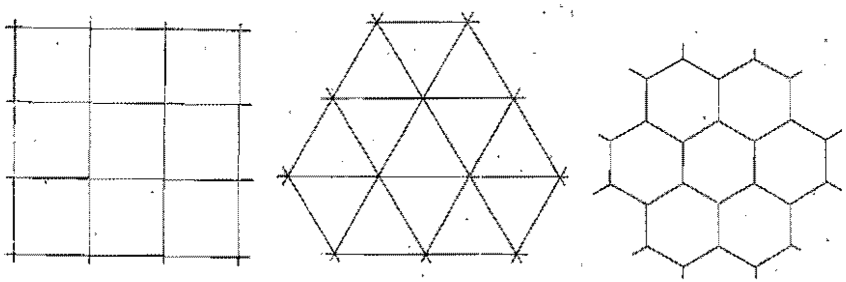


Fig. 4/4

In order to complete the description of the three groups in question we have to determine their translations.

Let $PQSR$ be a generating parallelogram of the triad-lattice obtained in the case $p=q=3$, PQR being an equilateral triangle (Fig. 4/5). The rotation through $2\pi/3$ about Q composed with

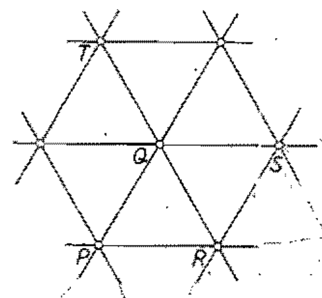
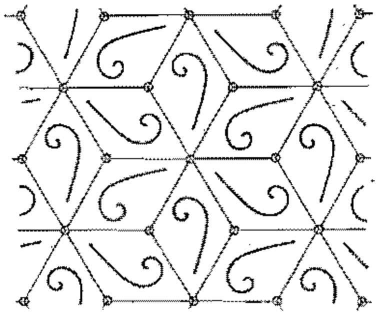


Fig. 4/5

the rotation through $4\pi/3$ about S results in the translation $P \to S$. This is the smallest translation of the group. Since translations other than those transforming a triad into a triad are out of the question, we have only to consider, say, the translation $P \to R$. This translation followed by a third-turn about R transforms PR into RQ, showing that the resulting motion would be a third-turn about the centre of PQR. This contradicts our previous result, according to which the lattice points are the only centres of rotation.

If PST is an equilateral triangle, then the translation $P \to T$ also belongs to the group. The two non-parallel translations $P \to S$ and $P \to T$, being the smallest ones contained in the group, generate the entire subgroup of translations of the group.

Recapitulating: the group $\mathfrak{W}_3$ so constructed consists of the translations and the positive and negative third-turns leaving the tessellation $\{3, 6\}$ or $\{6, 3\}$ invariant. The triads are, in both cases, the face-centres and vertices of the tessellation. A suitable unit cell is one of the three rhombi into which a face of $\{6, 3\}$ can be decomposed (Fig. 4/6).



$\mathfrak{W}_3$
Fig. 4/6

Turning now to the case $p = q = 4$, let P be a tetrad and Q_1, Q_2, Q_3, Q_4 the nearest tetrads forming the vertices of the square $Q_1Q_2Q_3Q_4$ (Fig. 4/7). Since, for instance, the translation $P \to Q_1$

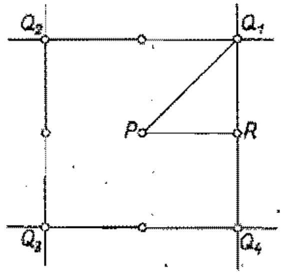
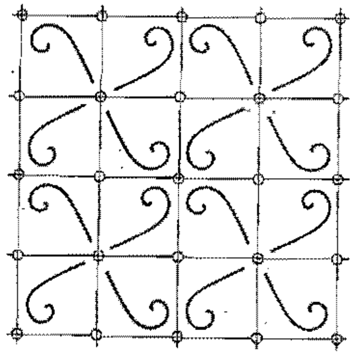


Fig. 4/7

together with the quarter-turn about Q would furnish the, inadmissible quarter-turn about the midpoint of the side Q_1Q_2. the smallest admissible translations are $Q_1 \to Q_2$ and $Q_2 \to Q_3$f These translations really belong to the group as the products of the half-turns about a vertex of the square and the midpoint o the subsequent side. Thus the square $Q_1Q_2Q_3Q_4$ is a generating



$\mathfrak{W}_4$
Fig. 4/8

parallelogram of the subgroup of translations, whereby the group is completely specified. This group, $\mathfrak{W}_4$, is the rotation group of the tessellation $\{4, 4\}$ (Fig. 4/8). The simplest unit cell is the triangle PQ_1Q_2.

Coming to the last case $p = 3$, $q = 6$, let P be a triad and Q_1, Q_2, Q_3 the nearest hexads (Fig. 4/9). Together with the half-

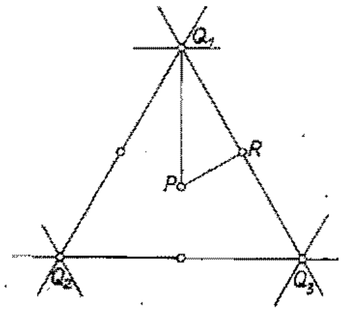
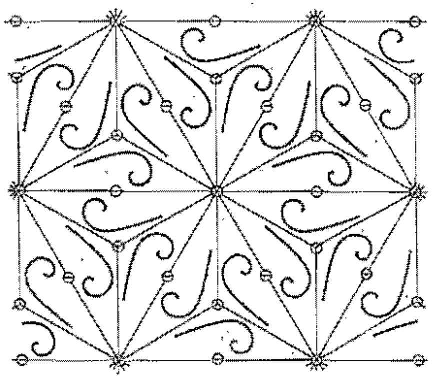


Fig. 4/9

turns about the vertices and the midpoints of the sides of the triangle $Q_1Q_2Q_3$, the translations $Q_1 \to Q_2$ and $Q_1 \to Q_3$ must also belong to the group. But since a translation of the group displaces one hexad into another, there are no smaller translations in the group, showing that the above-mentioned translations account for all translations of the group. This group, $\mathfrak{W}_6$, is the rotation group of $\{3, 6\}$ or $\{6, 3\}$ (Fig. 4/10). The triangle PQ_1Q_2 is a unit cell of the group.



$\mathfrak{W}_6$
Fig. 4/10

It is interesting to observe that, in spite of the greater variety of different rotations, $\mathfrak{W}_6$ admits of a somewhat simpler discussion than $\mathfrak{W}_3$ and $\mathfrak{W}_4$: there was no doubt about the diads (as by $\mathfrak{W}_4$), nor about the generating translations. But it is the very diversity of the rotations which facilitates the treatment.

We have now realized our first aim. We have constructed all crystallographic groups of proper motions in the plane, laying the foundation for our subsequent discussions.

First of all, we attempt to enlarge a group of type $\mathfrak{W}_1$ by a reflection R in a line l. Let a smallest translation of the group displace a point A of l into B and let B' be the image of B under R (Fig. 4/11). Then the group contains also the translation $A \to B'$. If A, B and B' are not collinear then the translations $A \to B$ and $A \to B'$ generate the entire set of translations of the group.

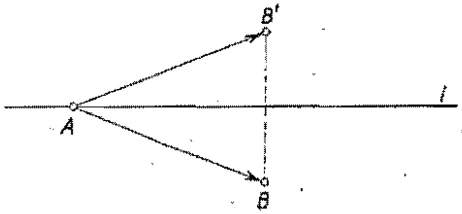


Fig. 4/11

In the opposite case, i.e. if AB is parallel or perpendicular to l, let a smallest translation of the group non-parallel to AB move A into C (Fig. 4/12). To determine the position of the point C, we consider the strip S bounded by the orthogonal bisector b of the segment AB and the parallel line a passing through A. Clearly, we may suppose, without loss of generality, that C lies in or on the boundary of S. If C is an inner point of S, we reflect C in l, obtaining C'. But then the composition of the translations $A \to C$ and $A \to C'$ or $A \to C$ and $C' \to A$ would yield a translation of the group smaller than the translation $A \to B$. This contradiction shows that C can lie only on a or b. Accordingly the translation group is generated either by the translations $A \to B$ and $A \to C$ or $A \to C$ and $A \to C'$. Therefore we have, in each case, as generating parallelogram either

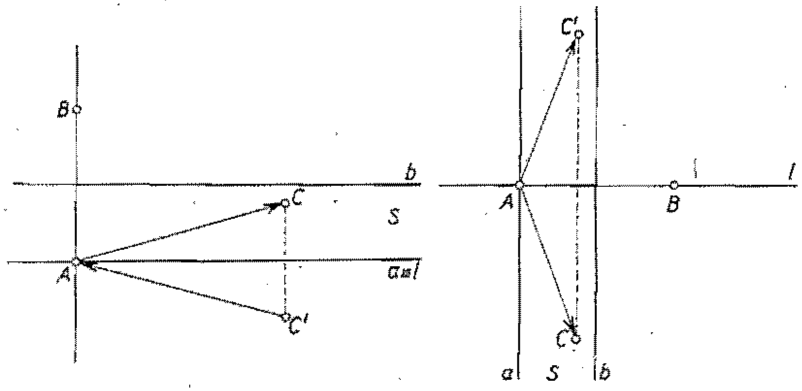
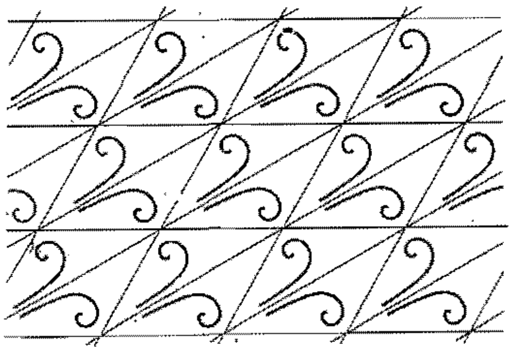


Fig. 4/12

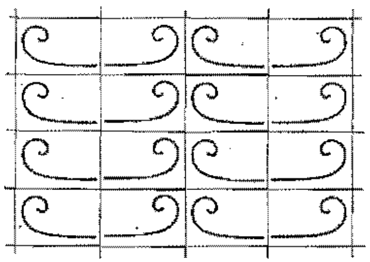
a rhombus or a rectangle, the mirror line being parallel to one diagonal of the rhombus or to one side of the rectangle, respectively. This gives rise to two groups, $\mathfrak{W}_1^1$ and $\mathfrak{W}_1^2$, the first being generated by two non-parallel translations through equal distances and a reflection in a line which bisects the angle formed by the directions of the translations (Fig. 4/13), the second one



$\mathfrak{W}_1^1$
Fig. 4/13

being generated by two orthogonal translations and a reflection in a line parallel to one translation (Fig. 4/14).

The transforms of one diagonal of the generating rhombus under its own lattice group unite in a set of parallel lines forming the mirror lines of the group $\mathfrak{W}_1^1$. The distance between two consecutive mirror lines equals half the length of the other



$\mathfrak{W}_1^2$
Fig. 4/14

diagonal, in accordance with the fact that the reflections in these lines generate a subgroup $\mathfrak{F}_1^2$ of $\mathfrak{W}_1^1$. A glance at Fig. 4/13 shows that there are also axes of glide-reflection, viz. the medians between consecutive axes of reflection. One of the triangles into which the first considered diagonal divides the rhombus is a unit cell.

Also $\mathfrak{W}_1^2$ contains a set of parallel axes of reflection but does not contain axes of glide-reflection. Of course, the group

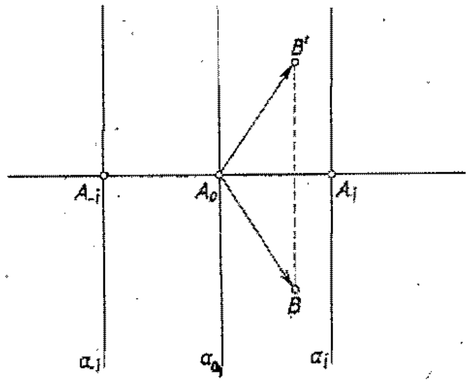


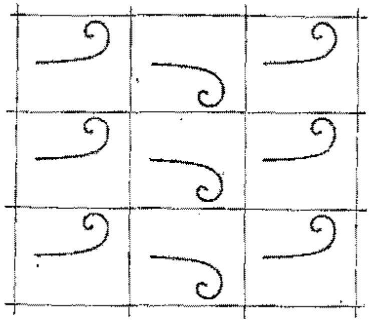
Fig. 4/15

contains glide-reflections, but only in lines which are simultaneously common mirror lines. As unit cell we can use a rectangle lying between two consecutive mirror lines.

We shall now show that a glide-reflection cannot be built into a lattice group other than a rhombic or rectangular one. Let G be a glide-reflection in the line l contained in a wall-pattern group $\mathfrak{W}$. The smallest subgroup which contains G and the translations parallel to l must be either $\mathfrak{F}_1^1$ or $\mathfrak{F}_1^3$. Since the first case, involving a reflection, has already been settled, we may suppose that G^2 is the shortest translation parallel to l.

Let a_0 be a line perpendicular to l cutting it in the point A_0 and let a_i and A_i be the images of a_0 and A_0 under G^i (Fig. 4/15). Further, let a translation T of $\mathfrak{W}$ displace A_0 into B. Then $G^{-1}TG$ is a translation too, namely the translation $A_0 \to B'$,

B' being the image of B under the reflection in l. This shows that B can only lie on a line a_l. For, if B lay between a_0 and a_1, the translations $A_0 \to B$ and $A_0 \to B'$ would produce a translation parallel to l smaller than A_0A_2, in contradiction to our stipulation according to which G^2 is the smallest translation parallel to l. Thus we can find, in accordance with our assertion, either a rhombus of diagonal A_0A_2 or a rectangle based upon A_0A_2 such that they are generating parallelograms of the lattice formed by the transforms of A_0 under the translations of $\mathfrak{W}$.



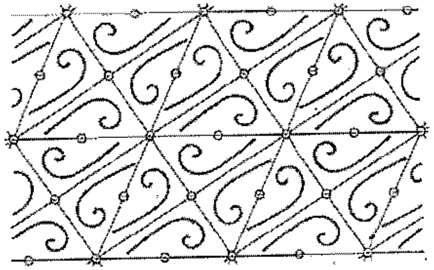
$\mathfrak{W}_1^3$
Fig. 4/16

The rhombic case having been settled, as the case of $\mathfrak{W}_1^1$, we obtain one further group, $\mathfrak{W}_1^3$, generated by a glide-reflection and a translation orthogonal to the glide axis (Fig. 4/16). Thus we have a set of parallel glide axes, the distance between adjacent axes being the half distance of the generating translation.

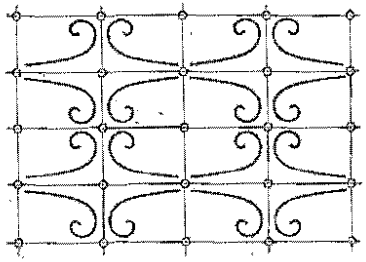
Turning now to the groups containing rotations, the condition of the invariance of the centres of rotation, together with the previous ones concerning the subgroup of translations, automatically cover all possibilities.

First we consider the groups in which, apart from diads, no other centres of rotation are present. Then the diads constitute a point lattice homothetic (similar and similarly situated) to the lattice generated by the subgroup of translations, having, in proportion to this, half the linear dimensions. This lattice must be either rhombic or rectangular.

To begin with, let the lattice of the diads be rhombic and suppose that the group contains a reflection in a line. Owing to the condition of invariance of the lattice under the reflection, this line must pass through a diad parallel to one of the diagonals of the generating rhombus. In this way the group, $\mathfrak{W}_2^1$, is com-



$\mathfrak{W}_2^1$
Fig. 4/17

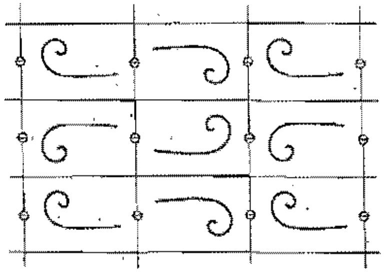


$\mathfrak{W}_2^3$
Fig. 4/18

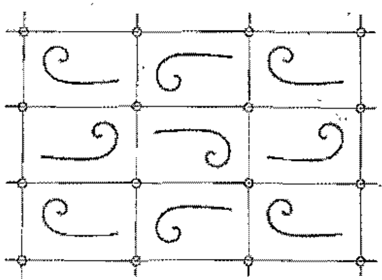
pletely determined (Fig. 4/17). It contains two sets of parallel mirror lines, both interlocked with a set of glide lines. The diagonals of a rhombus generating the subgroup of translations decompose it into four right triangles, one of them being a unit cell of the group.

Since in a group with a rhombic subgroup of translations the axes of glide-reflection necessarily interlock with axes of reflection, the rhombic case furnishes only the one group $\mathfrak{W}_2^1$ which has just been dealt with. On the other hand, a rectangular lattice of diads gives rise to three further groups $\mathfrak{W}_2^2$, $\mathfrak{W}_2^3$ and $\mathfrak{W}_2^4$.

The group $\mathfrak{W}_2^2$ arises by enlarging a rectangular group $\mathfrak{W}_2$ by a reflection in a line passing through a diad (Fig. 4/18). There are two families of parallel axes of reflection. A generating rectangle of the diad lattice is a unit cell.



$\mathfrak{W}_2^3$
Fig. 4/19



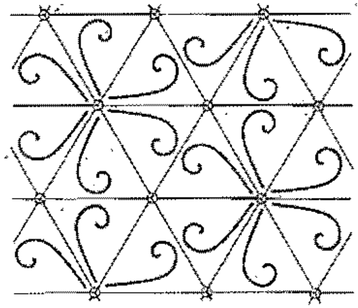
$\mathfrak{W}_2^4$
Fig. 4/20

If the line of reflection does not contain diads, it can only be the median between two neighbouring rows of diads, furnishing the group $\mathfrak{W}_2^3$ (Fig. 4/19). There is a set of parallel axes of reflection and an orthogonal set of axes of glide-reflection through the diads. These two sets of axes divide the plane into rectangles, one of which forms a unit cell.

Since a glide-reflection in a line passing through a diad involves a reflection in an orthogonal line, we have but one further possibility: a glide axis in the central line of two neighbouring rows of diads. The group $\mathfrak{W}_2^4$ arising in this way contains two sets of parallel glide lines (Fig. 4/20). As unit cell we can use a rectangle generating the lattice of the diads.

We now turn to the groups arising from $\mathfrak{W}_3$. Since $\mathfrak{W}_3$ contains as subgroup a rhombic lattice group, reflections and glide reflections always present themselves simultaneously. Hence we need to consider reflections only.

Let us consider a rhombus generating the lattice of the triads. We have two possibilities for enlarging $\mathfrak{W}_3$ by a reflection,



$\mathfrak{W}_3^1$
Fig. 4/21

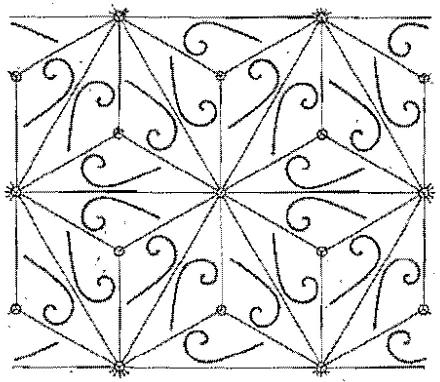
according as the mirror line contains the shorter diagonal or the longer one. Accordingly, we obtain the groups $\mathfrak{W}_3^1$ and $\mathfrak{W}_3^2$.

$\mathfrak{W}_3^1$ contains three families of parallel axes of reflection, making together the totality of the edges of a $\{3, 6\}$ (Fig. 4/21). The triads are the vertices of this $\{3, 6\}$, and one face of it is a unit cell.

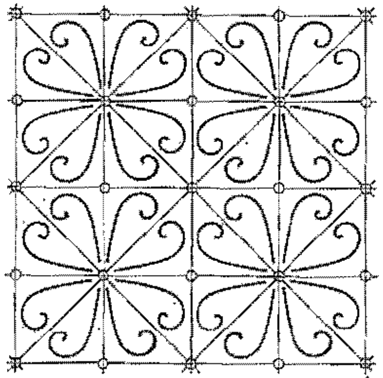
Also the set of the mirror lines of $\mathfrak{W}_3^2$ consists of the edges of a $\{3, 6\}$ (Fig. 4/22). But here the triads are the vertices and the face centres. The triangle determined by two vertices and the centre of a face is a unit cell.

Let us now consider a generating square of the lattice group of $\mathfrak{W}_4$. A glide-reflection in a line parallel to one side involves, in view of the presence of tetrads, a reflection or glide-reflection in a line parallel to one diagonal. This enables us to consider reflections only. As a result of the two possibilities depending on whether the mirror lines contain or do not contain tetrads, we have the groups $\mathfrak{W}_4^1$ and $\mathfrak{W}_4^2$.

$\mathfrak{W}_4^1$ is the symmetry group of the tessellation $\{4, 4\}$ (Fig. 4/23). It contains four families of parallel axes of reflection, but only two families of glide-reflection axes, parallel to the diagonals



$\mathfrak{W}_3^2$
Fig. 4/22

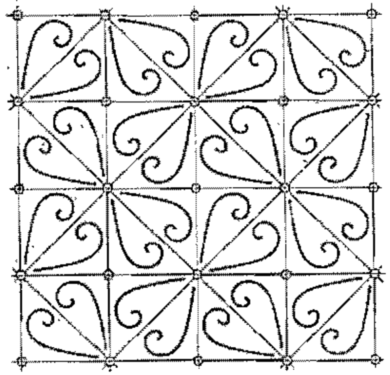


$\mathfrak{W}_4^1$
Fig. 4/23

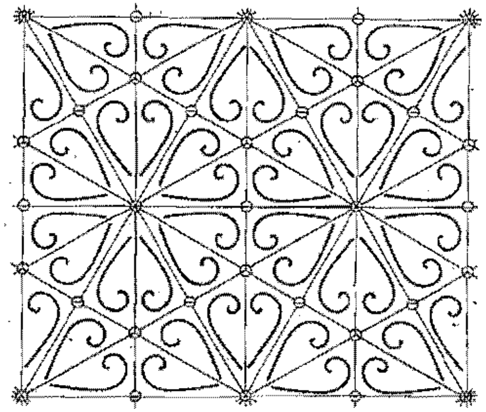
of the faces of $\{4, 4\}$. Its unit cell is an isosceles right triangle having as vertices two of the nearest tetrads and a diad.

If the axes of reflection do not contain tetrads each of them must pass through two of the nearest diads. Thus $\mathfrak{W}_4^2$ contains two

sets of axes of reflection (Fig. 4/24). The isosceles right triangle, having as vertices two of the nearest diads and a tetrad, yields a unit cell.



$\mathfrak{W}_4^2$
Fig. 4/24



$\mathfrak{W}_6^1$
Fig. 4/25

Last but not least (in view of the riches of its symmetry operations) we derive from $\mathfrak{W}_6$ the single group $\mathfrak{W}_6^1$ (Fig. 4/25). For, as above, we must not regard glide-reflections separately, and the axes of reflection are uniquely determined by the principle of invariance of the centres of rotations. In each hexad there cross six mirror lines. This is the symmetry group of the tessellation $\{3, 6\}$ or $\{6, 3\}$. Its unit cell is a triangle determined by a diad, triad and hexad of the least mutual distance from one another.

To sum up, there are 17 wall-pattern groups of different structure.

5. Remarks

The investigation of isometries goes back to Euclid. The fact that every proper motion of the plane can be achieved by a rotation seems to have been first recognized by Euler, who proved the analogous theorem for the sphere in 1776. The theory of congruent transformations was developed by Chasles, Rodrigues, Cayley, Sylvester, Hamilton and Donkin. In particular, the rotation-product theorem of Section 1 is due to Sylvester.

The foundation of the theory of groups was laid by Lagrange, the Italian physician Ruffini, Abel, Galois and Cauchy. The concept of the symmetry group occurs with Möbius and Hessel. Weyl (1952) ascribes the complete enumeration of the finite ornamental groups $\mathfrak{c}_p$ and $\mathfrak{d}_p$ to Leonardo da Vinci. It seems that the symmetry group $\mathfrak{d}_p$ is more frequent in art, nature and everyday life than $\mathfrak{c}_p$, rotatory symmetry being generally accompanied by bilateral symmetry. The first direct mathematical treatment of the 17 two-dimensional crystallographic groups was given by Fedorov in 1891 a few months after the analogous 3-dimensional groups involving the 2-dimensional ones. They were rediscovered by Fricke and Klein (1897), Pólya (1924) and Niggli (1924). The present treatment follows, in some respects, that of Hilbert and Cohn-Vossen (1932) and of Coxeter (1948).

Many of these groups were discovered empirically by the ancient Egyptians and the Chinese in their decorations. All 17 of them were known to the Moors, as shown by the ornaments decorating the walls of the Alhambra in Granada. This intuitive discovery of the ornamental groups is considered by Speiser (1958) as one of the greatest mathematical achievements of ancient times. In a similar spirit Weyl (1952) writes: "One can hardly overestimate the depth of geometric imagination and inventiveness reflected in these patterns. Their construction is far from being mathematically trivial. The art of ornament contains in implicit form the oldest piece of higher mathematics known to us."

The ornaments on Plates I—III are selected from the magnificent books of O. Jones, *The Grammar of Ornament,* London 1856 and A. Racinet, *L'Ornement Polychrome,* Paris 1869.

On the Greek friezes I/1,2,3 we recognize some principles on which the honeysuckle grows without finding any attempt at imitation. The softly curved lines reflect grace and noble simplicity. The symmetry group of the friezes I/1 and I/2 is $\mathfrak{F}_2^2$ and that of I/3 is $\mathfrak{F}^2$.

Egyptians used in their ornaments S- and C-shaped volutes, their native flowers and other elements which all had a symbolic meaning. The lotus and papyrus symbolized the carnal and intellectual food; the beetle on I/4 was used as an emblem of immortality. The symmetry groups of the Egyptian wall-patterns I/4—7 are $\mathfrak{W}_1^1$, $\mathfrak{W}_2^3$, $\mathfrak{W}_2^1$ and $\mathfrak{W}_4$. Ignoring the difference of the colours in the ornament I/6 we obtain the group $\mathfrak{W}_4^2$.

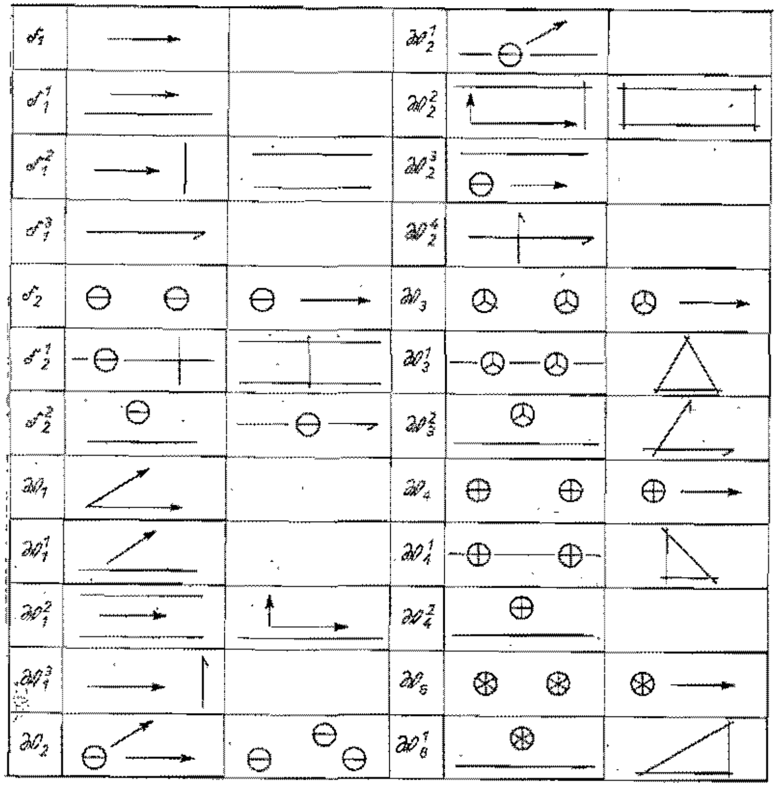
The artful drawings II/1,2 are of Chinese origin. They illustrate the groups $\mathfrak{W}_3^1$ and $\mathfrak{W}_4^2$. In II/2 the swastikas are centred in the tetrads.

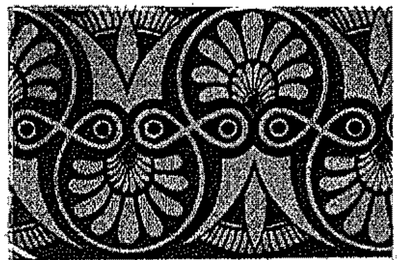
II/3 represents a fine piece of the Alhambra with the symmetry group $\mathfrak{W}_6$. Though perfect in itself, the absence of bilateral symmetry suggests a kind of turbulence. In contrast to the Egyptians the religion of the Moors forbade symbolism in art. Banishing emblematic figures they carried the treatment of strict geometrical form to the highest degree of refinement and elegance. Blue, red and yellow (gold) are typical and exclusive colours employed on Moorish stucco-works.

The group $\mathfrak{W}_6^1$ is illustrated by Plate III representing a gilded oak ceiling of the early Renaissance in the Palais de Justice in Rouen. It spreads repose and perfection.

Extremely ingenious wall-patterns may be found in Escher (1960). The significance of symmetry is illustrated by various examples in nature or art in Jaeger (1917), Thompson (1952), Nicolle (1950), Speiser (1952, 1958), Weyl (1952), Wolf and Wolff (1956), Coxeter, Opechowski and Wright (1957) and Coxeter (1961).

Certain elements of a group constitute a set of *generators* if every element of the group may be expressed as a finite product of their (positive and negative) powers. In order to survey the $7 + 17$ infinite ornamental groups we have tabulated one or two sets of generators for these groups by the aid of a suggestive symbolism. For instance, $\mathfrak{F}_1$ is generated by a translation, $\mathfrak{F}_1^3$ by a glide-reflection, $\mathfrak{W}_3^1$ by two third-turns and a reflection in a line passing through the centres of rotation or by reflections in three lines bounding an equilateral triangle.



1


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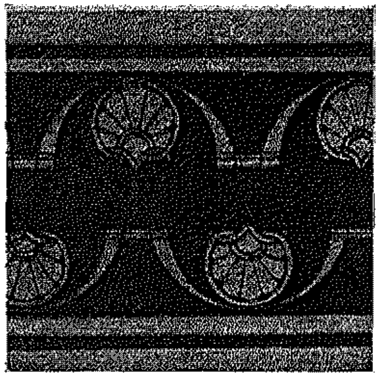

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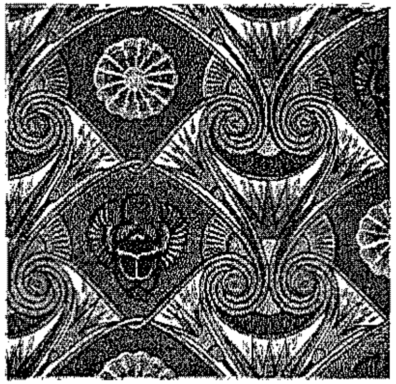

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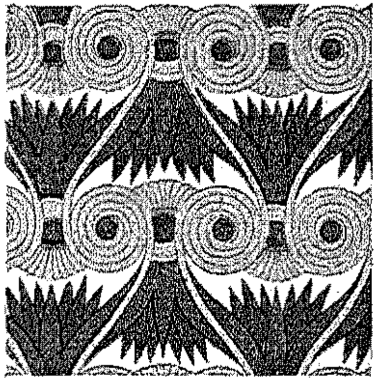

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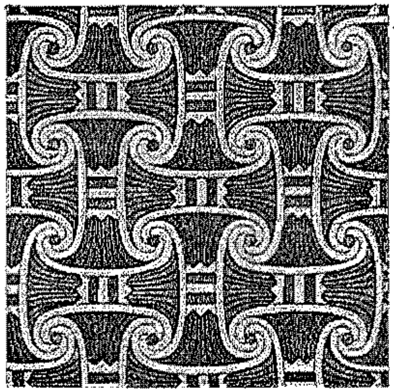

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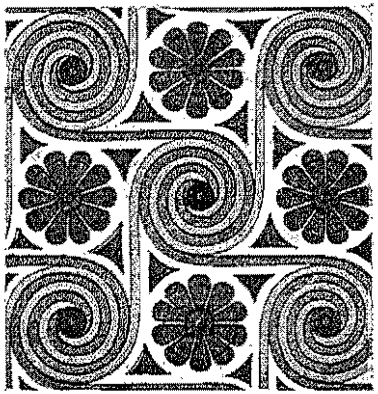
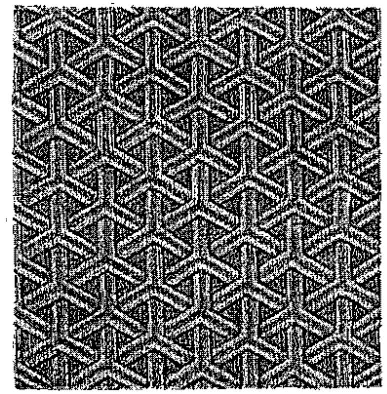
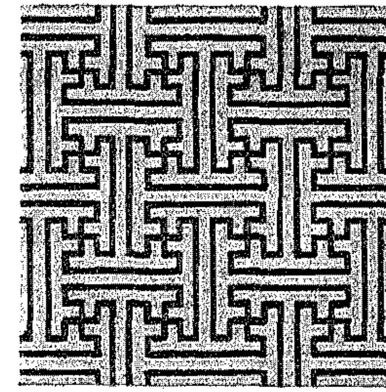


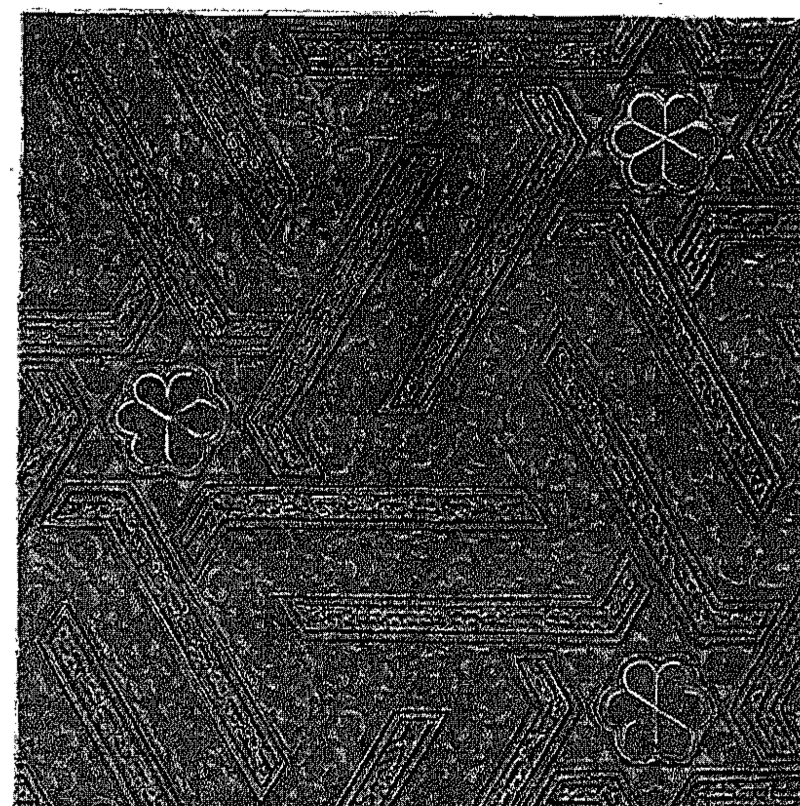
Plate I

1





2



3

Plate II

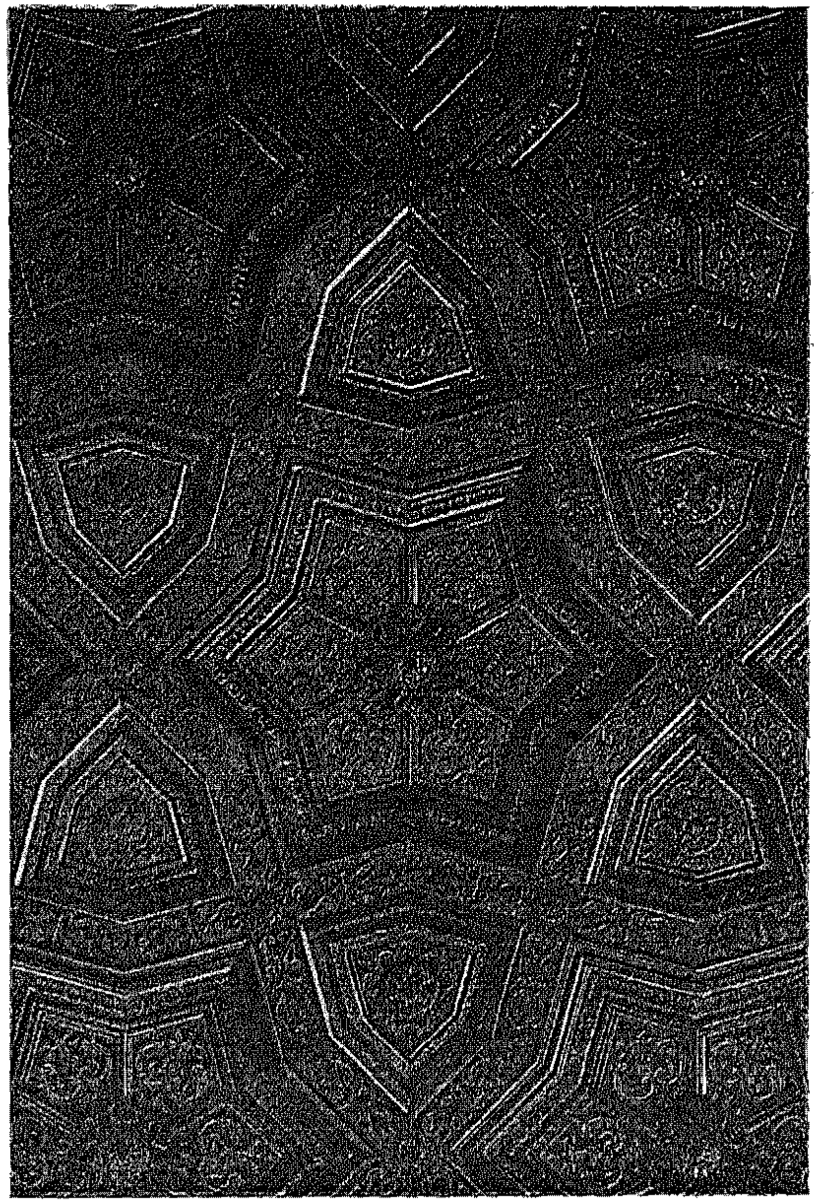


Plate III

A set of relations satisfied by the generators of a group is called an *abstract definition* of the group if every relation satisfied by the generators is an algebraic consequence of these particular relations. Nowacki (1954) showed the 17 wall-pattern groups to be abstractly distinct. (Note that $\mathfrak{F}_1$ and $\mathfrak{F}_1^2$ are abstractly identical.) Coxeter and Moser (1957) have given abstract definitions for these groups. Their definition of $\mathfrak{W}_4^1$ is

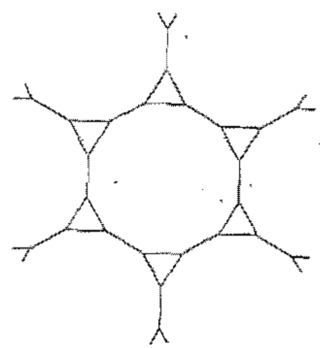
$$R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^4 = (R_2R_3)^4 = (R_3R_1)^2 = 1$$

and that of $\mathfrak{W}_6^1$

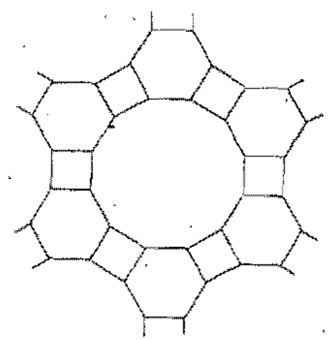
$$R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^3 = (R_2R_3)^6 = (R_3R_1)^2 = 1\,.$$

They pointed out that these two groups comprise all infinite ornamental groups as subgroups and gave a complete table of the subgroup relationships among the wall-pattern groups. Many of these relationships had already been indicated by Niggli (1924).

The groups $\mathfrak{d}_p$, $\mathfrak{F}_1^2$, $\mathfrak{F}_2^1$, $\mathfrak{W}_2^2$, $\mathfrak{W}_3^1$, $\mathfrak{W}_4^1$ and $\mathfrak{W}_6^1$ deserve special attention, in so far as they can be generated by reflections alone. Following Möbius, these groups can be illustrated by a "kaleidoscope" consisting of a suitable horizontal region, the unit cell of the group, bounded by vertical mirrors which represent the generating reflections. The theory of the analogous groups in general spaces of constant curvature was developed by Goursat, E. Cartan, Coxeter, Wythoff, Witt, Weyl and others.

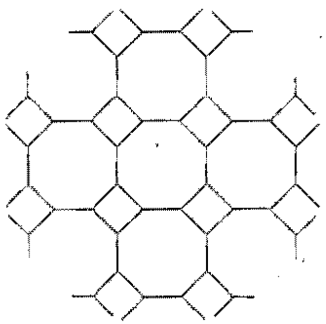


(3, 12, 12)

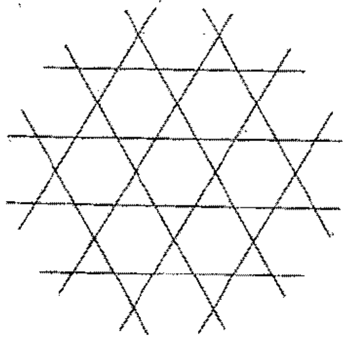


(4, 6, 12)

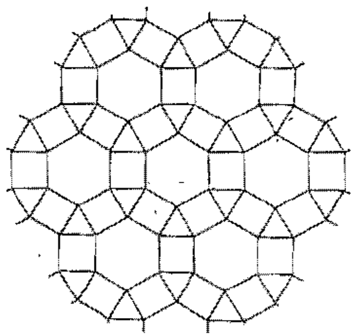
Fig. 5/1



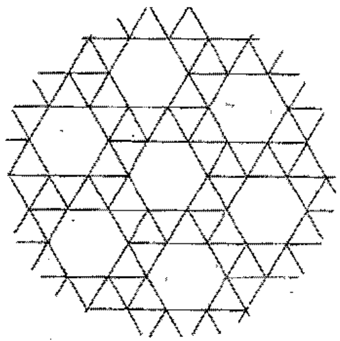
(4, 8, 8)



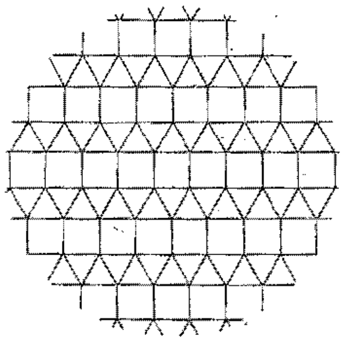
(3, 6, 3, 6)



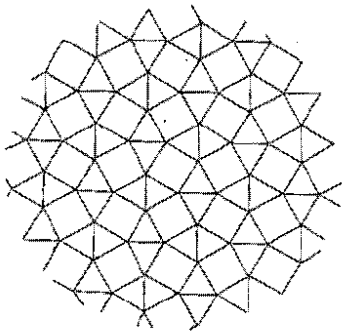
(3, 4, 6, 4)



(3, 3, 3, 3, 6)



(3, 3, 3, 4, 4)



(3, 3, 4, 3, 4)

Fig. 5/1

The symbol $\{p, q\}$ for a regular tessellation is a special case of the general Schläfli symbol (to be discussed later) for a regular decomposition of a general space of constant curvature. Besides the three regular tessellations $\{3, 6\}$, $\{4, 4\}$ and $\{6, 3\}$ there are in the Euclidean plane eight *semiregular* (Archimedean) *tessellations* having incongruent regular faces and equivalent vertices.

Denoting such a tessellation by a symbol giving the number of sides of the faces surrounding a vertex (in their proper cyclic order), the tessellations in question are (3, 12, 12), (4, 6, 12), (4, 8, 8), (3, 4, 6, 4), (3, 6, 3, 6), (3, 3, 3, 3, 6), (3, 3, 3, 4, 4) and (3, 3, 4, 3, 4) (Fig. 5/1). With the exception of (3, 3, 3, 4, 4), they may all be constructed by different kinds of "truncations" of the regular tessellations (Fig. 5/2). These tessellations may be approached from a more general point of view. Research into the structure of crystals involves the consideration of non-overlapping equal spheres arranged regularly in space, and similar arrangements of circles in the plane. However, the metrical properties of such circle-packings are by no means given by their symmetry groups. For, by varying the centre of a circle in the unit cell, the transforms of the circle under the same symmetry group will yield circle-packings entirely different from the standpoint of crystallography. (Consider, for example, the circle-packings represented by Fig. 5/3.1 and 5/3.28, which have $\mathfrak{W}_3^1$ as common symmetry groups.) This necessitates a finer classification of the regular circle-packings (and sphere-packings).

A useful notion for this purpose is the *inner group* of a circle-packing. This is the group of all symmetry operations of the packing which leave a circle invariant. The whole symmetry group may then be called the *outer group* of the packing. Making use of the researches of Sohncke, Barlow, Niggli and others, Sinogowitz (1938) divided — on the basis of the inner and outer group — the totality of the regular circle-packings into thirty-one classes. In this classification group properties still prevail. A further distinction, emphasizing more the metrical properties, can be obtained by the aid of the *Dirichlet cell*, which consists